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A Sum Rule for the Two-Dimensional Two-Component Plasma

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In a two-dimensional, two-component plasma, the second moment of the *density* correlation function has the simple value $\{12\pi[1-(\Gamma/4)]^2\}^{-1}$, where Γ is the dimensionless coupling constant. This result is derived by using analogies with critical systems.

KEY WORDS: Coulomb systems; sum rule; critical behavior.

1. INTRODUCTION AND SUMMARY

The system under consideration is the two-dimensional two-component plasma (Coulomb gas), i.e. a system of positive and negative point-particles of charge $\pm q$, in a plane. Two particles at a distance r from each other interact through the two-dimensional Coulomb interaction $\mp q^2 \ln(r/L)$, where L is some irrelevant length. Classical equilibrium statistical mechanics is used. The dimensionless coupling constant is $\Gamma = q^2/kT$, where k is Boltzmann's constant and T the temperature. The system of point charges is known to be stable against collapse when $\Gamma < 2$ and then to have the simple exact equation of state[(1)]

$$p = kTn\left(1 - \frac{\Gamma}{4}\right) \tag{1.1}$$

where p is the pressure and n the total number density of the particles.

This paper is dedicated to George Stell.

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Let $\langle \hat{n}(0)\, \hat{n}(\mathbf{r}) \rangle_T$ be the truncated density-density two-point function (correlation function), where $\hat{n}(\mathbf{r})$ is the total microscopic number density at $\mathbf{r}$. The sum rule which is claimed here is

$$\int \langle \hat{n}(0)\, \hat{n}(\mathbf{r}) \rangle^T r^2 d^2\mathbf{r} = \frac{1}{12\pi(1-\Gamma/4)^2} \tag{1.2}$$

This sum rule is about the second moment of the *number* density correlation function and should not be confused with the well-known Stillinger–Lovett[2] sum rule obeyed by the charge correlation function

$$\int \langle \hat{\rho}(0)\, \hat{\rho}(\mathbf{r}) \rangle\, r^2 d^2\mathbf{r} = -\frac{2kT}{\pi} \tag{1.3}$$

where $\hat{\rho}(\mathbf{r})$ is the microscopic *charge* density.

Unfortunately, I have not been able to derive (1.2) by an argument direct enough for my taste (perhaps the publication of the present paper will trigger somebody producing such an argument). In Section 2, a rather indirect argument is given. In Section 3, the sum rule is tested in two special cases. In Section 4, another indirect derivation is made.

2. FROM SPHERE TO PLANE

In the present Section, the sum rule (1.2) for the system living in a plane is derived from known properties of that system living on the surface of a sphere.

2.1. The System on a Sphere

On the surface of a sphere of radius R, the two-dimensional Coulomb interaction[3] between two particles i and j at an angular distance ψ_{ij} from each other is $\mp q^2 \ln[(2R/L) \sin(\psi_{ij}/2)]$. Indeed, the corresponding total electric potential is a solution of Poisson's equation, provided the total charge is zero, a condition which will be assumed.

The Bolzmann factor for two particles is $[(2R/L) \sin(\psi_{ij}/2)|^{\mp\Gamma}$ and the grand partition function, restricted to zero total charge, is

$$\Xi = 1 + \lambda^2 R^4 \int \frac{d\Omega_1\, d\Omega_2}{(2R/L \sin \psi_{12}/2)^{\Gamma}} + \cdots \tag{2.1}$$

where λ is the fugacity and $d\Omega_i$ is an element of solid angle around the position of particle i. The last explicitly written term of (2.1) involves 1 positive and 1 negative particles. It is easily seen that, more generally, the term of (2.1) involving N positive and N negative particles is $(\lambda^2 L^\Gamma R^{4-\Gamma})^N$ times a dimensionless integral depending on Γ. Therefore, $\ln \Xi$ depends on λ and R only through the combination $\lambda^2 R^{4-\Gamma}$ and obeys the homogeneity relation

$$\lambda \frac{\partial}{\partial \lambda} \ln \Xi = \frac{1}{2} \left(1 - \frac{\Gamma}{4}\right)^{-1} R \frac{\partial}{\partial R} \ln \Xi \tag{2.2}$$

where the left-hand side is the total number of particles, i.e. $4\pi R^2$ times the total number density n_S on the sphere:

$$4\pi R^2 n_S = \lambda \frac{\partial}{\partial \lambda} \ln \Xi \tag{2.3}$$

For a given fugacity λ, in the large-R limit, one should recover a plane system with pressure p. $\ln \Xi$ must be extensive, behaving like $(p/kT)\, 4\pi R^2$, and from (2.2) and (2.3) one obtains the equation of state (1.1).

The key ingredient of the present argument is that, for a large but not yet infinite value of R, there is a universal finite-size correction[4] to $\ln \Xi$, similar to the one which occurs in a system with short-range forces at its critical point: At a fixed fugacity, the large-R expansion of $\ln \Xi$ starts as

$$\ln \Xi = \frac{p}{kT} 4\pi R^2 - \frac{1}{3} \ln R + \text{constant} + \cdots \tag{2.4}$$

Using (2.2) and (2.3) in (2.4), one obtains the finite-size correction to the number density, for a given fugacity:

$$n_S = n - \frac{1}{24\pi R^2 (1 - \Gamma/4)} + \cdots \tag{2.5}$$

where n_S is the number density for the system on a large sphere of radius R and n the number density for the plane system.

2.2. Stereographic Projection

By a suitable stereographic projection, the two-component plasma on a sphere can be mapped onto a modified two-component plasma on a plane.

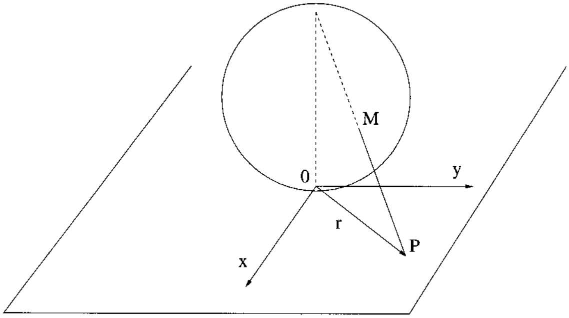

Fig. 1. The stereographic projection.

Let P be the stereographic projection of a point M of the sphere of radius R onto the plane tangent to its South pole, from its North pole (Fig. 1). Let $\mathbf{r} = (x, y)$ be the Cartesian coordinates of P. An area element $R^2\, d\Omega$ around M and its projection $d^2\mathbf{r}$ on the plane are related by

$$R^2\, d\Omega = \frac{d^2\mathbf{r}}{(1 + r^2/4R^2)^2} \tag{2.6}$$

The angular distance ψ_{12} between two points M_1 and M_2 on the sphere is related to the distance $|\mathbf{r}_1 - \mathbf{r}_2|$ between their projections on the plane by

$$2R \sin \frac{\psi_{12}}{2} = \frac{|\mathbf{r}_1 - \mathbf{r}_2|}{(1 + r_1^2/4R^2)^{1/2}\,(1 + r_2^2/4R^2)^{1/2}} \tag{2.7}$$

Written in terms of the plane coordinates $\mathbf{r}_i$, the grand partition function (2.1) on the sphere becomes

$$\Xi = 1 + \lambda^2 \int \left(\frac{L}{|\mathbf{r}_1 - \mathbf{r}_2|}\right)^{\Gamma} \frac{d^2\mathbf{r}_1}{(1 + r_1^2/4R^2)^{2-(\Gamma/2)}} \frac{d^2\mathbf{r}_2}{(1 + r_2^2/4R^2)^{2-(\Gamma/2)}} + \cdots \tag{2.8}$$

By the same change of variables in the general term of (2.1), it can be seen that (2.8) is the grand partition function of a modified plane two-component plasma: in addition to the two-body interactions $\mp q^2 \ln(|\mathbf{r}_i - \mathbf{r}_j|/L)$, there is a sign-independent one-body potential $V(r_i)$ acting on each particle:

$$\frac{1}{kT} V(r) = 2\left(1 - \frac{\Gamma}{4}\right) \ln\left(1 + \frac{r^2}{4R^2}\right) \tag{2.9}$$

2.3. Density Shift and Linear Response

A density shift is caused by the additional potential (2.9). In the large-R limit, (2.9) can be replaced by

$$\frac{1}{kT} V(r) = \left(1 - \frac{\Gamma}{4}\right) \frac{r^2}{2R^2} \tag{2.10}$$

and linear response theory gives for the density shift of the plane system at the origin 0

$$\delta n(0) = -\frac{1}{kT} \int \langle \hat{n}(0)\, \hat{n}(\mathbf{r}) \rangle_T\, V(r)\, d^2\mathbf{r} \tag{2.11}$$

where the statistical average in the right-hand side is to be taken in the unperturbed system, i.e. the plane system considered in Section 1.

On the other hand, since the density n_S on the sphere and its projection $n(\mathbf{r})$ on the plane are equal at the South pole 0,

$$\delta n(0) = n_S - n \tag{2.12}$$

as given by (2.5).

From (2.5), (2.10), (2.11), (2.12), one obtains (1.2).

3. TESTS

The sum rule (1.2) can be tested when $\Gamma = 2$ or when Γ is small.

3.1. The Case $\Gamma = 2$

At $\Gamma = 2$, the two-dimensional two-component plasma is exactly solvable[5–7] by a mapping onto a free fermion field. The density correlation function is

$$\langle \hat{n}(0)\, \hat{n}(\mathbf{r}) \rangle_T = 2 \left(\frac{m^2}{2\pi}\right)^2 [-K_0^2(mr) + K_1^2(mr)] + n\delta(\mathbf{r}) \tag{3.1}$$

where m is the rescaled fugacity $m = 2\pi\lambda L$ and K_0 and K_1 are modified Bessel functions. (3.1) does obey the sum rule (1.2), with $\Gamma = 2$.

3.2. The Case of Small Γ

When Γ is small, the Debye approach gives[8] a screened effective potential $K_0(\kappa r)$ where κ is the inverse Debye length: $\kappa^2 = 2\pi n q^2/kT = 2\pi n\Gamma$.

An approximate form of the density correlation function is

$$\langle \hat{n}(0)\,\hat{n}(\mathbf{r})\rangle_T = \tfrac{1}{2}\, n^2\big(e^{\Gamma K_0(\kappa r)} + e^{-\Gamma K_0(\kappa r)} - 2\big) + n\delta(\mathbf{r})$$

or, equivalently since $\Gamma \ll 1$, the pair correlation function is

$$h(r) = \tfrac{1}{2}\,\Gamma^2 K_0^2(\kappa r) \tag{3.2}$$

This approximate correlation function passes the test of the compressibility sum rule

$$n \int h(r)\, d^2\mathbf{r} = kT \frac{\partial n}{\partial p} - 1 \tag{3.3}$$

since the left-hand side of (3.3) is $\Gamma/4$ when (3.2) is used, while the right-hand side of (3.3) is also $\Gamma/4$ at order Γ from (1.1).

When used in the left-hand side of the sum rule (1.2), (3.2) gives $1/12\pi$, which is the correct result when $\Gamma \to 0$.

4. THE MASSIVE THIRRING MODEL

The sum rule (1.2) can be related to another, already known, sum rule obeyed by a field-theoretical model: the massive Thirring model. This model is described in terms of a two-component field ψ of Dirac fermions by the Euclidean action

$$S = -\int \left[\bar{\psi}(\partial\!\!\!/ + m_0)\,\psi + g(\bar{\psi}\psi)^2\right] d^2\mathbf{r} \tag{4.1}$$

When the coupling constant g does not vanish, the bare mass m_0 is renormalized into m.

The massive Thirring model should obey the sum rule[9]

$$\int \langle \mathscr{E}(0)\,\mathscr{E}(\mathbf{r})\rangle_T\, r^2 d^2\mathbf{r} = \frac{1}{3\pi m^2 (2-\Delta)^2} \tag{4.2}$$

where[2] $\mathscr{E}(\mathbf{r}) = \bar{\psi}\psi$ and $\Delta = [1 + (g/\pi)]^{-1}$. This sum rule (4.2) is an application to the Thirring model of a very general formula of Cardy[10] about almost-critical two-dimensional systems. In the case of the Thirring model, (4.2) has been explicitly checked, to first order in g, by Naón.[9]

[2] Ref. 9 has i factors which do not appear in the present Euclidean formalism.

The two-component plasma can be mapped[11] onto the massive Thirring model, with the correspondences

$$\frac{\Gamma}{2} = \Delta, \qquad \hat{n}(\mathbf{r}) = m\mathscr{E}(\mathbf{r}) \tag{4.3}$$

(the special case $\Gamma = 2$ corresponds to $g = 0$, i.e. a free-field theory, and this is why the two-component plasma is exactly solvable at $\Gamma = 2$). The correspondence (4.3) makes the sum rules (1.2) and (4.2) identical.

5. CONCLUSION

The sum rule (1.2) has been derived through analogies with the theory of critical phenomena. A more direct derivation is still wanted.[3]

As a final remark, it should be noted that the two-component plasma of point particles is stable only for $\Gamma < 2$. However, at $\Gamma = 2$, although the density n diverges (for a given finite fugacity), the correlation function $\langle \hat{n}(0)\, \hat{n}(\mathbf{r}) \rangle_T$ is finite (for $r \neq 0$). It is tempting to conjecture that $\langle \hat{n}(0)\, \hat{n}(\mathbf{r}) \rangle_T$ remains finite and that the sum rule (1.2) holds in the whole range $0 < \Gamma < 4$. This conjecture is supported by a similar statement[10] about the sum rule (4.2).

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REFERENCES

1. E. H. Hauge and P. C. Hemmer, *Phys. Norv.* **5**:209 (1971).
2. F. H. Stillinger and R. Lovett, *J. Chem. Phys.* **49**:1991 (1968).
3. J. M. Caillol, *J. Physique Lettres* **42**:L-245 (1981).
4. B. Jancovici, G. Manificat, and C. Pisani, *J. Stat. Phys.* **76**:307 (1994).
5. M. Gaudin, *J. Physique* **46**:1027 (1985).
6. F. Cornu and B. Jancovici, *J. Stat. Phys.* **49**:33 (1987).
7. F. Cornu and B. Jancovici, *J. Chem. Phys.* **90**:2444 (1989).
8. C. Deutsch and M. Lavaud, *Phys. Rev. A* **9**:2598 (1974).
9. C. M. Naón, *J. Phys. A* **22**:2877 (1989).
10. J. L. Cardy, *Phys. Rev. Lett.* **60**:2709 (1988).
11. See e.g. J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena* (Clarendon, Oxford, 1989).
12. B. Jancovici, P. Kalinay, and L. Šamaj, *Physica A* **279**:260 (2000).

[3] This more direct derivation has now been obtained.[12]